The computable model theory of uncountably categorical models (Part III)

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More Classical Model Theory

Definition:

1. An infinite definable subset $\varphi(\mathfrak{M})$ of a model $\mathfrak{M}$ is strongly minimal if any definable subset of $\varphi(\mathfrak{M}')$ in any elementary extension $\mathfrak{M}'$ of $\mathfrak{M}$ is finite or cofinite.
2. A model $\mathfrak{M}$ is strongly minimal if any definable subset of any elementary extension $\mathfrak{M}'$ of $\mathfrak{M}$ is finite or cofinite.

(Here "definable" = "definable with parameters".)

Remark:

Strong minimality (of a model) implies uncountable categoricity and is a property of the theory.

Baldwin-Lachlan Theorem (1971, contd.):

Each uncountably categorical model contains a strongly minimal subset over which it is the prime model. The dimension of the model is (roughly) the size of the largest algebraically independent subset.

More Classical Model Theory (contd.)

Recall:

1. The <u>algebraic closure of a set</u> $A \subseteq M$ is the set of all $m \in M$ which are the contained in a finite set definable over A. (Such $m \in M$ is called <u>algebraic over</u> A.)
2. If $m \in M$ is algebraic over A then m is algebraic over a <u>finite</u> subset $A' \subseteq A$.
3. A theory T is <u>model complete</u> if for any models $\mathfrak{M} \subset \mathfrak{N}$ of T, we have $\mathfrak{M} \prec \mathfrak{N}$.
4. If a theory T is model complete, then the set $T_{\forall\exists}$ of its $\forall\exists$-consequences <u>axiomatizes</u> T.

Definition:

Let $\mathfrak{M}$ be a strongly minimal model.

1. Then M together with the algebraic closure operator forms a pregeometry, i.e., $\mathrm{acl}(\text{-})$ is a finitary closure operator with the exchange property.
2. This pregeometry is trivial if for all nonempty subsets $A \subseteq M$,

$$\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(\{a\})$$

Remark:

All the above-mentioned uncountably categorical models are strongly minimal with trivial pregeometry.

Theorem

(Goncharov, Harizanov, Laskowski, Lempp, McCoy)

For any trivial, strongly minimal theory T, the elementary diagram $\mathrm{Th}(\mathfrak{M}_M)$ of $\mathfrak{M}$ is a model complete L_M-theory (i.e., in the expansion by constants for all elements of M).

Corollary:

Let $\mathfrak{M}$ be computable, trivial, strongly minimal model.
Then $\mathrm{Th}(\mathfrak{M})$ forms a $\mathbf{0}''$-computable theory.
Thus all countable models of $\mathrm{Th}(\mathfrak{M})$ are $\mathbf{0}''$-decidable (and so in particular $\mathbf{0}''$-computable).

Proof:
By the Theorem, $\mathrm{Th}_{\forall\exists}(\mathfrak{M}_M)$ axiomatizes $\mathrm{Th}(\mathfrak{M}_M)$, so the latter, and a fortiori $\mathrm{Th}(\mathfrak{M})$,
is a $\mathbf{0}''$-computable set.
Now by Harrington/Khisamiev (relativized to $\mathbf{0}''$), each countable models of $\mathrm{Th}(\mathfrak{M})$ is $\mathbf{0}''$-decidable (and so in particular $\mathbf{0}''$-computable).

Remark:

By an example of Goncharov and Khoussainov, the assumption of strong minimality in the above corollary is necessary.

A first computability-theoretic proof attempt for the corollary:

Define an "infinitary" logic L^∞ by replacing the usual first-order quantifiers by

- $\forall^\infty$ ("for all but finitely many"), and
- $\exists^{<\infty}$ ("there exist at most finitely many").

Proposition:

The L^∞-theory (indeed the L^∞-elementary diagram) of any strongly minimal model is $\mathbf{0}'$-computable.

Proof: Use induction on the number of free variables, querying oracle $\mathbf{0}'$ repeatedly, since

- $\forall^\infty$ is equivalent to $\exists^{\leq k}\neg$ and $\exists^{>k}$ (for some k), and
- $\exists^{<\infty}$ is equivalent to $\exists^{\leq k}$ and $\exists^{>k}\neg$ (for some k).

by the following lemma (and we can find the appropriate k computably in $\mathbf{0}'$).

Nonfinite Covering Property Lemma:

For any strongly minimal model $\mathfrak{M}$ and any formula $\varphi(\bar{x}, \bar{y})$, there is a finite bound k such that for any $\bar{b} \in M$, $\varphi(\mathfrak{M}, \bar{b})$ is infinite or has size at most k.

Claim:

For any L^{∞}-formula $\varphi(\bar{x}, \bar{y})$ and any $\bar{b} \in M$, the set $\{\bar{a} : \mathfrak{M} \models \varphi(\bar{a}, \bar{b})\}$ is computable, with index uniformly computable in $\mathbf{0}'$.

Proof:

By induction on the quantifier complexity of $\varphi(\bar{x}, \bar{y})$:
Fix $\varphi(\bar{x}, \bar{y}) \equiv \exists z\, \psi(\bar{x}, \bar{y}, z)$.
For $k = 1, 2, \ldots$, check, using $\mathbf{0}'$, if there are

- k many distinct $c \in M$ with $\mathfrak{M} \models \psi(\bar{a}, \bar{b}, c)$, and
- k many distinct $d \in M$ with $\mathfrak{M} \models \neg\psi(\bar{a}, \bar{b}, c)$.

One of these will eventually fail by the above lemma.

More on Model Completeness (Kueker):

Definition:

An $\forall\exists$-formula $\theta(\overline{y})$ and an existential formula $\psi(\overline{x},\overline{y})$ (both in L) form a <u>linked pair</u> (for T) if

1. $T \models \exists\overline{y}\theta(\overline{y})$, and
2. $T \models \forall\overline{y}\forall\overline{y'}(\theta(\overline{y}) \wedge \theta(\overline{y'}) \wedge \psi(\overline{x},\overline{y}) \rightarrow \psi(\overline{x},\overline{y'}))$

Proposition:

$\mathrm{Th}(\mathfrak{M}_M)$ is model complete iff for each L-formula $\varphi(\overline{x})$, there is a linked pair (θ,ψ) (for $\mathrm{Th}(\mathfrak{M})$) such that

$$\mathfrak{M} \models \forall\overline{y}(\theta(\overline{y}) \rightarrow \forall\overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x},\overline{y}))).$$

(Thus model completeness of the elementary diagram is a property of the theory!)

Model-theoretic Corollary:

Every trivial, strongly minimal theory is $\exists\forall\exists$-axiomatizable.

Computability-theoretic Corollary:

The spectrum of computable models $\mathrm{SCT}(T)$ of a trivial, strongly minimal (and not totally categorical) theory T is a Σ^0_5-subset of $\omega \cup \{\omega\}$.

Remarks:

1. The only examples known thus far of spectra are intervals, and only one kind of these is neither an initial nor a final segment of $\omega \cup \{\omega\}$.
2. On the other hand, no arithmetical bound for the complexity of $\mathrm{SCT}(T)$ was previously known.

Proof of the Theorem:

We fix a model $\mathfrak{M}_0$ of T and set

$$T^* = \mathrm{Th}((\mathfrak{M}_0)_{M_0})$$

$$L^* = L_{M_0}$$

We need to show T^* to be model complete,
i.e., that given $\mathfrak{M}_0 \prec \mathfrak{M}, \mathfrak{N}$ and $\mathfrak{M} \subset \mathfrak{N}$,
we have $\mathfrak{M} \prec \mathfrak{N}$.
(In fact, it suffices to show this for $\mathfrak{M}$, $\mathfrak{N}$ of some fixed cardinality $> |M_0|$.)

Definition:

Call an L^*-formula $\varphi(\bar{x})$ <u>absolute</u> if for all $\bar{b} \in M$,

$$\mathfrak{M} \models \varphi(\bar{b}) \text{ iff } \mathfrak{N} \models \varphi(\bar{b})$$

We need to show that every L^*-formula is absolute.

Call $\varphi(\bar{x},\bar{y})$ an (n,m)-<u>formula</u>

if $\bar{x},\bar{y}$ are of length n, m, respectively.

Define the following statements:

$A_{n,m}$: For all absolute (n,m)-formulas $\varphi(\bar{x},\bar{y})$,

$$\exists^{<\infty}\bar{y}\varphi(\bar{x},\bar{y}) \text{ is absolute.}$$

$B_{n,m}$: For all absolute (n,m)-formulas $\varphi(\bar{x},\bar{y})$

and all $\bar{b} \in M$,

$$\mathfrak{N} \models \exists^{<\infty}\bar{y}\ \varphi(\bar{b},\bar{y}) \quad \text{implies} \quad \varphi(\bar{b},\mathfrak{M}) = \varphi(\bar{b},\mathfrak{N}).$$

$C_{n,m}$: For all absolute (n,m)-formulas $\varphi(\bar{x},\bar{y})$,

$$\exists\bar{y}\varphi(\bar{x},\bar{y}) \quad \text{is absolute.}$$

The absolute formulas are closed under Boolean connectives, so we need to show $C_{n,1}$ for all n.

Claims:

1. $A_{n,m} \Rightarrow A_{n',m'}$ whenever $n \geq n'$ and $m \geq m'$
   (and similarly for $B_{n,m}$ and $C_{n,m}$).
2. $B_{n,m} \Rightarrow C_{n,m}$
3. $B_{n,m} \Rightarrow A_{n,m+1}$
4. $B_{1,m}$ holds
5. $B_{n,m+1} \wedge A_{n+1,m} \Rightarrow B_{n+1,m}$

Now $C_{n,1}$ follows by induction on the claims.

Proof sketches:

1. Trivial.
3. Use $C_{n,m}$ (from 2.) to reduce $A_{n,m+1}$ to $B_{n,m}$.
4. Use strong minimality.

Proof:

2. Typical for the proofs of claims 3 and 4 also:
By induction on $k \le m$, prove $C_{n,k}$.
Note that $C_{n,0}$ is vacuous.
Assume $C_{n,k}$ for $k < m$.
Fix an absolute $(n, k+1)$-formula $\varphi(\bar{x}, \bar{y})$
and $\bar{b} \in M$ with $\mathfrak{N} \models \exists \bar{y}\, \varphi(\bar{b}, \bar{y})$.

Case 1: $\mathfrak{N} \models \exists^{<\infty} \bar{y}\, \varphi(\bar{b}, \bar{y})$: Use $B_{n,k+1}$ and so
$\varphi(\bar{b}, \mathfrak{M}) = \varphi(\bar{b}, \mathfrak{N})$.

Case 2: $\mathfrak{N} \models \exists^{\infty} \bar{y}\, \varphi(\bar{b}, \bar{y})$: Partition $\bar{y}$ into $w\bar{z}$
such that $\mathfrak{N} \models \exists^{\infty} w \exists \bar{z}\, \varphi(\bar{b}, w\bar{z})$.
Then $\{e \in N : \mathfrak{N} \models \exists \bar{z}\, \varphi(\bar{b}, e\bar{z})\}$ is cofinite,
so there is $a \in M_0$ such that

$$\mathfrak{N} \models \exists \bar{z}\, \varphi(\bar{b}, a\bar{z}).$$

Then $\varphi(\bar{b}, a\bar{z})$ is an absolute (n, k)-formula.
Now apply $C_{n,k}$.

Proof sketch:

5. Core of the argument:
Here we finally use triviality and the following fact from stability theory:

Finite Satisfiability Lemma:

Suppose $\mathfrak{M}_0 \prec \mathfrak{N}$ are strongly minimal and $\bar{b}, \bar{c} \in N$ such that

1. at least one of $\bar{b}, \bar{c}$ is in $\mathrm{acl}(M_0 \cup \{e\})$ for some single $e \in N$, and
2. $\mathrm{acl}(M_0 \cup \{\bar{b}\}) \cap \mathrm{acl}(M_0 \cup \{\bar{c}\}) = M_0$.

Then for any L^*-formula $\varphi(\bar{x}, \bar{y})$ and any $\bar{b}, \bar{c} \in N$, $\mathfrak{N} \models \varphi(\bar{b}, \bar{c})$ implies $\mathfrak{N} \models \varphi(\bar{a}, \bar{c})$ for some $\bar{a} \in M_0$.

Rough Sketch:

Fix an $(n,m+1)$-formula $\varphi(\bar{x},y,\bar{z})$ and $\bar{a},b\in M$ with $\mathfrak{N}\models\exists^{<\infty}\bar{z}\,\varphi(\bar{a},b\bar{z})$.

Fix $\bar{c}\in N$ with $\mathfrak{N}\models\varphi(\bar{a},b\bar{c})$.

We claim that $\bar{c}\in M$.

Fix $e\in M\setminus\mathrm{acl}(M_0\cup\{\bar{a}\})$.

Case 1: $\mathfrak{N}\models\exists^{\infty}\bar{z}\,\varphi(\bar{a},e\bar{z})$: Then there are only finitely many $d\in N$ with $\mathfrak{N}\models\exists^{<\infty}\bar{z}\,\varphi(\bar{a},d\bar{z})$.

By $A_{n+1,m}$,

$$\psi(\bar{x}y,\bar{z})\equiv\varphi(\bar{x}y,\bar{z})\wedge\exists^{<\infty}\bar{w}\,\varphi(\bar{x}y,\bar{w})$$

is absolute, and by the above

$$\mathfrak{N}\models\exists^{<\infty}y\bar{z}\,\psi(\bar{a},y\bar{z}).$$

Thus, by $B_{n,m+1}$, $b\bar{c}\in M$.

Case 2: $\mathfrak{N} \models \exists^{<\infty} \bar{z}\, \varphi(\bar{a}, e\bar{z})$: Let $\overline{c_j}\,(j \le r)$ be all the solutions $\bar{z}$, partition each $\overline{c_j}$ into $\overline{d_j}\,\overline{e_j}$ where $\overline{d_j} \in \mathrm{acl}(M_0 \cup \{\bar{a}\})$, $\overline{e_j} \in N \setminus \mathrm{acl}(M_0 \cup \{\bar{a}\})$.
By ***triviality***, $\overline{e_j} \in \mathrm{acl}(M_0 \cup \{e\})$.

By finite satisfiability,
$$\mathrm{acl}(M_0 \cup \{\bar{a}\overline{d_j}\}) \cap \mathrm{acl}(M_0 \cup \{e\overline{e_j}\}) = M_0.$$

By finite satisfiability (twice), we obtain

1. $b'\overline{e'_j} \in M_0$ with
$\mathfrak{N} \models \varphi(\bar{a}, b'_j \overline{d_j}\,\overline{e'_j}) \wedge \exists^{<\infty} \overline{u_j}\varphi(\bar{a}, b'_j \overline{u_j}\,\overline{e'_j})$ and

2. $\overline{a'}\,\overline{d'_j} \in M_0$ with
$\mathfrak{N} \models \varphi(\overline{a'}, b_j \overline{d'_j}\,\overline{e_j}) \wedge \exists^{<\infty} \overline{v_j}\varphi(\overline{a'}, b_j \overline{d'_j}\,\overline{v_j})$.
By $A_{n,m}$, both these formulas are absolute.
By similar further argument, $B_{n,m+1}$ shows $\bar{c} \in M$.

Possible Extensions

1. Does our result extend to strongly minimal models with locally modular nontrivial pregeometry? (By work of Hrushovski, any such model is essentially a vector space or an affine space over a division ring.)
2. Does our result generalize to non-strongly minimal uncountably categorical models with trivial pregeometry? (There is a good model-theoretic analysis of how the uncountably categorical model is built up from the strongly minimal subset. By the example of Goncharov and Khoussainov, the result does not extend directly. However, recent ongoing work of Dolich, Laskowski and Raichev gives an arithmetical bound, probably depending only on the Morley rank.)
3. Does our result generalize to models of finite Morley rank? (Recall that any uncountably categorical model has finite Morley rank.)
4. Can $\mathbf{0}''$ be improved to $\mathbf{0}'$? We conjecture "no", and can prove "no" for the uniform version.