The computable model theory of uncountably categorical models (Part II)

Steffen Lempp

(http://www.math.wisc.edu/~lempp)

University of Wisconsin–Madison

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Recall from Classical Model Theory

Definition:
A first-order theory T is λ-categorical if T has only one model of power λ (up to isomorphism).

Assume now: The first-order language L is countable.

Morley's Theorem (1965):
If a first-order theory T is categorical in some uncountable power, then T is categorical in all uncountable powers. (Such a theory is called uncountably categorical.)

Baldwin-Lachlan Theorem (1971):
The countable models of an uncountably categorical, not totally categorical theory form an elementary chain $\mathfrak{M}_0 \prec \mathfrak{M}_1 \prec \cdots \prec \mathfrak{M}_\omega$
(where $\mathfrak{M}_0$ is the prime model and $\mathfrak{M}_\omega$ is the countably saturated model).

Recall from
Computable Model Theory

Assume from now on:
The first-order language L is computable (recursive), i.e., L is countable; if L contains infinitely many relation or function symbols then in addition the arity of each relation and function symbol is computable (uniformly in the index of the relation or function symbol).

Definitions:
1. A countable model $\mathfrak{M}$ is computable (recursive, constructivizable) if it is isomorphic to a model $\mathfrak{M}'$ (with universe ω) such that the open diagram of $\mathfrak{M}'$ (the set of all quantifier-free $\mathfrak{M}'$-formulas) is computable. ($\mathfrak{M}'$ is called a presentation of $\mathfrak{M}$.)
2. A countable model $\mathfrak{M}$ is decidable (strongly constructivizable) if the elementary diagram of $\mathfrak{M}'$ (the set of all $\mathfrak{M}'$-formulas) is computable.
3. Relativizing to any set X, we can define the notions of X-computable and X-decidable.

Two Questions (Lempp, mid-1990's):

- Do the above results on spectra of computable models necessarily require an infinite language?

- If some model of an uncountably categorical first-order theory is computable, what can we say about its other countable models?
  Must the other countable models be
  - arithmetical?
  - $\mathbf{0}''$-computable?
  - $\mathbf{0}'$-computable?

A Related Question (Nies, Shore):

- How complicated can the spectrum of computable models be?

(Nies observed that a trivial upper bound is $\Sigma^0_{\omega+3}$.)

Theorem (Herwig, Lempp, Ziegler, 1999):

There is an uncountably categorical theory in one binary relation such that $\mathrm{SCM}(T) = \{0\}$, i.e., only the prime model is computable.

Remarks:
1. Any uncountably categorical theory in only finitely many unary relations is totally categorical, so the above result is best possible.
2. We can make (the open diagrams of) the nonprime models of any given Δ^0_2-degree.

Proof:
The model "codes" the "Cayley graph" of a finitely generated group with unsolvable word problem such that the word problem can be computed only from the nonprime models.
We show how to establish the result using three binary relations. (One can easily code three binary relations by one single binary relation.)
Our proof uses a group-theoretic lemma:

Group-theoretic Lemma:

Let F be the free group of rank 3, generated by a, b, t, say. Then there is a sequence $\{N_k\}_{k\in\omega}$ of subgroups of F such that:

1. $\{N_k\}_{k\in\omega}$ is a uniformly computable sequence of normal subgroups of F of finite index; thus each F / N_k is finite and has solvable word problem (uniformly in k).
2. For each $w \in F$, the set $\{k \in \omega : w \in N_k\}$ is either finite or cofinite; thus the pointwise limit
$$N = \{w \in F : \{k \in \omega\} \text{ is cofinite}\}$$
of the N_k exists and is a normal subgroup of F.
3. N is noncomputable; thus F / N has unsolvable word problem.

Remark:

We can make the word problem of F / N of any given Δ_2^0-degree.

Proof of Theorem:

The Cayley graph $\mathfrak{C}_k$ of F/N_k has finite universe

$$C_k = F/N_k$$

and three binary relations R_a, R_b, R_t defined by

$$R_x(v,w) \text{ iff } v = w \cdot x \quad (\text{for } x \in \{a,b,t\}).$$

We now define the prime model $\mathfrak{M}_0$ to be the disjoint union of the (finite) Cayley graphs $\mathfrak{C}_k$.

Since each $\mathfrak{C}_k$ is computable (uniformly in k), $\mathfrak{M}_0$ is also computable (as a model in the three binary relations R_a, R_b, R_t).

Any nonprime model $\mathfrak{M}_\alpha$ contains also α many copies of the (infinite) Cayley graph $\mathfrak{C}$ of F/N, since the "balls" $B_r(g)$ of radius r, defined as the sets of all those elements connected to $g \in C_k$ by a sequence of at most r many R-edges, have a single fixed isomorphism type for sufficiently large k (depending on r).

But since F/N has unsolvable word problem, $\mathfrak{C}$ cannot be computable, and thus no nonprime model $\mathfrak{M}_\alpha$ can be computable.

Proof Sketch of the Group-theoretic Lemma:

Given $k > 0$, we define:

1. the symmetric group
$$S_3 = \langle a, \phi \mid a^3 = \phi^2 = a^{\phi} a = 1 \rangle$$
2. the wreath product
$$H_k = S_3 \text{ wr } \mathbf{Z}_{2k+1}$$
$$= \left\langle S_3, b \;\middle|\; \begin{array}{c} b^{2k+1}, [a^{b^j}, a^{b^{j'}}], [\phi^{b^j}, \phi^{b^{j'}}], [a^{b^j}, \phi^{b^{j'}}] \\ (-k \le j < j' \le k) \end{array} \right\rangle$$
3. the subgroup
$L_k = \langle a, b, t \rangle$ (here $t = \prod_{j \in K_k} \phi^{b^j}, K$ = halting problem)
4. the kernel N_k of the homomorphism of F onto L_k.

Then:

1. H_k is the semidirect product of $\bigotimes_{-k \le j \le k} S_3$ and $\mathbf{Z}_{2k+1}$, so all three groups are computable (uniformly in k); and
2. for all $w \in F, w \in N_k$ for finitely or cofinitely many k; and
3. $k \in A$ iff $[t, a^{b^k}] \in N$ for all k.

More Classical Model Theory

Definition:

1. An infinite definable subset $\varphi(\mathfrak{M})$ of a model $\mathfrak{M}$ is strongly minimal if any definable subset of $\varphi(\mathfrak{M}')$ in any elementary extension $\mathfrak{M}'$ of $\mathfrak{M}$ is finite or cofinite.
2. A model $\mathfrak{M}$ is strongly minimal if any definable subset of any elementary extension $\mathfrak{M}'$ of $\mathfrak{M}$ is finite or cofinite.

(Here "definable" = "definable with parameters".)

Remark: Strong minimality (of a model) implies uncountable categoricity, so is a property of the theory.

Baldwin-Lachlan Theorem (1971, contd.):

Each uncountably categorical model contains a strongly minimal subset over which it is the prime model. The dimension of the model is (roughly) the size of the largest algebraically independent subset. (Recall the examples of successor function on ω, vector spaces, and algebraically closed fields.)

More Classical Model Theory (contd.)

Recall:

3. The <u>algebraic closure of a set</u> $A \subseteq M$ is the set of all $m \in M$ which are the contained in a finite set definable over A. (Such $m \in M$ is called <u>algebraic over</u> A.)
4. If $m \in M$ is algebraic over A then m is algebraic over a <u>finite</u> subset $A' \subseteq A$.
5. A theory T is <u>model complete</u> if for any models $\mathfrak{M} \subset \mathfrak{N}$ of T, we have $\mathfrak{M} \prec \mathfrak{N}$.
6. If a theory T is model complete, then the set $T_{\forall\exists}$ of its $\forall\exists$-consequences <u>axiomatizes</u> T.

Recall again: Examples of successor function on ω, vector spaces, and algebraically closed fields.
(***Note***: Successor function <u>not</u> model complete without constant symbol for 0.)

Definition:

Let $\mathfrak{M}$ be a strongly minimal model.

1. Then M together with the algebraic closure operator forms a pregeometry, i.e., $\text{acl}(\text{-})$ is a finitary closure operator with the exchange property.
2. This pregeometry is trivial if for all nonempty subsets $A \subseteq M$,

$$\text{acl}(A) = \bigcup_{a \in A} \text{acl}(\{a\})$$

3. This pregeometry is locally modular if (roughly), for any algebraically closed $A, B \subseteq M$,

$$\dim(A \cup B) + \dim(A \cap B) = \dim(A) + \dim(B)$$

Remark:

All the above-mentioned uncountably categorical models are strongly minimal with trivial pregeometry.

Notation:

Given a model $\mathfrak{M}$ and a subset $X \subseteq M$, the expansion $\mathfrak{M}_X$ of $\mathfrak{M}$ by constants in X is obtained by adding constant symbols for each $x \in X$ (interpreted in the obvious way).
We denote the corresponding expansion of the language L by L_X.

Theorem

(Goncharov, Harizanov, Laskowski, Lempp, McCoy)

For any trivial, strongly minimal theory T,
the elementary diagram $\mathrm{Th}(\mathfrak{M}_M)$ of $\mathfrak{M}$
is a model complete L_M-theory.

Corollary:

Let $\mathfrak{M}$ be computable, trivial, strongly minimal model.
Then $\mathrm{Th}(\mathfrak{M})$ forms a $\mathbf{0}''$-computable theory.
Thus all countable models of $\mathrm{Th}(\mathfrak{M})$ are $\mathbf{0}''$-decidable (and so in particular $\mathbf{0}''$-computable).

***Proof*:**
By the Theorem, $\mathrm{Th}_{\forall\exists}(\mathfrak{M}_M)$ axiomatizes $\mathrm{Th}(\mathfrak{M}_M)$, so the latter, and a fortiori $\mathrm{Th}(\mathfrak{M})$,
is a $\mathbf{0}''$-computable set.
Now by Harrington/Khisamiev (relativized to $\mathbf{0}''$), each countable models of $\mathrm{Th}(\mathfrak{M})$ is
$\mathbf{0}''$-decidable (and so in particular $\mathbf{0}''$-computable).

Remark:

By an example of Goncharov and Khoussainov, the assumption of strong minimality in the above corollary is necessary.

A first computability-theoretic proof attempt for the corollary:

Define an "infinitary" logic L^{∞} by replacing the usual first-order quantifiers by

- $\forall^{\infty}$ ("for all but finitely many"), and
- $\exists^{<\infty}$ ("there exist at most finitely many").

Proposition:

The L^{∞}-theory (indeed the L^{∞}-elementary diagram) of any strongly minimal computable model is $\mathbf{0}'$-computable.

Proof: Use induction on the number of free variables, querying oracle $\mathbf{0}'$ repeatedly, since

- $\forall^{\infty}$ is equivalent to $\exists^{\leq k}\neg$ and $\exists^{>k}$ (for some k), and
- $\exists^{<\infty}$ is equivalent to $\exists^{\leq k}$ and $\exists^{>k}\neg$ (for some k).

by the following lemma (and we can find the appropriate k computably in $\mathbf{0}'$).

Nonfinite Covering Property Lemma:

For any strongly minimal model $\mathfrak{M}$ and any formula $\varphi(\overline{x},\overline{y})$, there is a finite bound k such that for any $\overline{b}\in M$, $\varphi(\mathfrak{M},\overline{b})$ is infinite or has size at most k.

Claim:
For any L^{∞}-formula $\varphi(\overline{x},\overline{y})$ and any $\overline{b}\in M$, the set $\{\overline{a}:\mathfrak{M}\models\varphi(\overline{a},\overline{b})\}$ is computable, with index uniformly computable in $\mathbf{0}'$.

Proof:
By induction on the quantifier complexity of $\varphi(\overline{x},\overline{y})$:
Fix $\varphi(\overline{x},\overline{y})\equiv\exists z\,\psi(\overline{x},\overline{y},z)$.
For $k=1,2,\ldots$, check, using $\mathbf{0}'$, if there are

- k many distinct $c\in M$ with $\mathfrak{M}\models\psi(\overline{a},\overline{b},c)$, and
- k many distinct $d\in M$ with $\mathfrak{M}\models\neg\psi(\overline{a},\overline{b},c)$.

One of these will eventually fail by the above lemma.

More on Model Completeness (Kueker):

Definition:

An $\forall\exists$-formula $\theta(\overline{y})$ and an existential formula $\psi(\overline{x},\overline{y})$ (both in L) form a linked pair (for T) if

1. $T \models \exists\overline{y}\theta(\overline{y})$, and
2. $T \models \forall\overline{y}\forall\overline{y'}(\theta(\overline{y}) \wedge \theta(\overline{y'}) \wedge$
   $\psi(\overline{x},\overline{y}) \rightarrow \psi(\overline{x},\overline{y'}))$

Proposition:

$\mathrm{Th}(\mathfrak{M}_M)$ is model complete iff for each L-formula $\varphi(\overline{x})$, there is a linked pair (θ,ψ) (for $\mathrm{Th}(\mathfrak{M})$) such that

$$\mathfrak{M} \models \forall\overline{y}(\theta(\overline{y}) \rightarrow \forall\overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x},\overline{y})).$$

(Thus model completeness of the elementary diagram is a property of the theory!)

Model-theoretic Corollary:

Every trivial, strongly minimal theory is $\exists\forall\exists$-axiomatizable.

Computability-theoretic Corollary:

The spectrum of computable models $\mathrm{SCT}(T)$ of a trivial, strongly minimal (and not totally categorical) theory T is a Σ^0_5-subset of $\omega \cup \{\omega\}$.

Remarks:

1. The only examples known thus far of spectra are intervals, and only one kind of these is neither an initial nor a final segment of $\omega \cup \{\omega\}$.
2. On the other hand, no arithmetical bound for the complexity of $\mathrm{SCT}(T)$ was previously known.