The computable model theory of uncountably categorical models (Part I)

Steffen Lempp

(http://www.math.wisc.edu/~lempp)

University of Wisconsin–Madison

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Overview of Lecture Series

- Motivation
- Definitions from classical model theory and from computable model theory
- Basic results used from classical model theory
- Spectrum of computable models (results of Goncharov, Kudaibergenov, Khoussainov / Nies / Shore, Herwig / Lempp / Ziegler, Nies, and Hirschfeldt / Nies)
- Computational complexity of noncomputable models (result of Goncharov / Harizanov / Laskowski / Lempp / McCoy)

Two Goals of Computable Model Theory

1. Determine the computational complexity of models of a given first-order theory, in particular whether the models can be represented effectively.

2. Compare the computational complexity of various models of a given first-order theory.

These goals are particularly relevant in the case where

- the first-order theory has "few" models, and

- the structure of the models of the theory is well-understood classically.

Thus this applies especially in the case of uncountably categorical models.

Classical Model Theory

Definition:
A first-order theory T is λ-categorical if T has only one model of power λ (up to isomorphism).

Assume from now on:
The first-order language L is countable.

Morley's Theorem (1965):
If a first-order theory T is categorical in some uncountable power, then T is categorical in all uncountable powers. (Such a theory is called uncountably categorical.)

Baldwin-Lachlan Theorem (1971):
The countable models of an uncountably categorical, but not totally categorical theory form an elementary chain

$$\mathfrak{M}_0 \prec \mathfrak{M}_1 \prec \cdots \prec \mathfrak{M}_\omega$$

(where $\mathfrak{M}_0$ is the prime model and $\mathfrak{M}_\omega$ is the countably saturated model).

The three standard examples:

- $\mathrm{Th}(\omega, S)$, the theory of the natural numbers with ***successor*** function: The κth model is
  $$M_\kappa = \omega \,\dot{\cup}\, \dot{\bigcup_{\alpha<\kappa}} Z.$$

- Theory of a ***vector space*** over the rationals, with scalar multiplication by each rational given by a separate unary function: The κth model is the vector space of dimension $1+\kappa$.

- Theory of ***algebraically closed fields*** of characteristic 0: The κth model is the algebraically closed field of characteristic 0 and transcendence degree κ.

Computable Model Theory

Assume from now on:
The first-order language L is computable (recursive), i.e., L is countable; if L contains infinitely many relation or function symbols then in addition the arity of each relation and function symbol is computable (uniformly in the index of the relation or function symbol).

Definitions:

1. A countable model $\mathfrak{M}$ is computable (recursive, constructivizable) if it is isomorphic to a model $\mathfrak{M}'$ (with universe ω) such that the open diagram of $\mathfrak{M}'$ (the set of all quantifier-free $\mathfrak{M}'$-formulas) is computable.
   ($\mathfrak{M}'$ is called a computable presentation of $\mathfrak{M}$.)
2. A countable model $\mathfrak{M}$ is decidable (strongly constructivizable) if the elementary diagram of $\mathfrak{M}'$ (the set of *all* $\mathfrak{M}'$-formulas) is computable.
3. Relativizing to any set X, we can define the notions of X-computable and X-decidable.

First Results in Computable Model Theory

Theorem (folklore):
A decidable first-order theory has a decidable model.
(Here a theory is deductively closed.)

Proof:
Effectivize the Henkin construction:
Given a decidable theory, its Henkinization is also decidable.
Since this result holds uniformly, the ωth iterate of the Henkinization is also decidable.
Now build a decidable model using that each element is named by a term and that equality of terms is decidable.

Question:
Must all countable models of a decidable theory be decidable?

Answer: No, but . . .

Theorem (Harrington, Khisamiev, 1974):
All countable models of a decidable uncountably categorical theory are decidable.
(By relativization, all countable models of an X -decidable uncountably categorical theory are X -decidable.)

Proof: By Baldwin-Lachlan (1971), each countable model of a decidable uncountably categorical theory T is the prime model of a decidable uncountably categorical expansion T' of T.
By Baldwin (1973), the rank function, assigning to each L'-formula its Morley rank (an integer) is computable.
Given any L'-formula φ, we can effectively find a computable principal type Γ_φ containing φ.
The prime model of T' can now be constructed by an effective Henkin argument and an effective omitting types argument.

Question: If a countable, uncountably categorical model is computable, must then all the other countable models be computable?

Answer: NO, leading to the following definition:

Spectrum of Computable Models

Definition:
The <u>spectrum of computable models</u> of an uncountably categorical theory T is the set
$\mathrm{SCM}(T) = \{\alpha \le \omega : \mathfrak{M}_\alpha \text{ is computable}\}$.
(From now on, we will always tacitly assume T is not totally categorical, i.e., not also ω-categorical.)

All Known Possible Spectra:

1. (e.g., Rabin, 1960) $\mathrm{SCM}(T) = \omega \cup \{\omega\}$ (using algebraically closed fields)
2. (Goncharov, 1978) $\mathrm{SCM}(T) = \{0\}$
3. (Kudaibergenov, 1980) $\mathrm{SCM}(T) = \{0, 1, \ldots, n\}$ (for any positive integer n)
4. (Khoussainov, Nies, Shore, 1997) $\mathrm{SCM}(T) = \omega$
5. (same authors, 1997) $\mathrm{SCM}(T) = \{1, 2, \ldots, \omega\}$
6. (Nies, 1999) $\mathrm{SCM}(T) = \{1\}$
7. (Hirschfeldt, Nies / Fokina, n.d.)
   $\mathrm{SCM}(T) = [1, \alpha)$ (for any $\alpha \le \omega$)

Proof Sketches:

We describe the language and the prime model first:

2. The language contains a binary relation R and unary relations U_s (for $s \in \omega$).
The prime model $\mathfrak{M}_0$ has universe ω with

$$R(x,s) \text{ iff } x \in K_s, \text{ and}$$

$$U_s = [s,\infty),$$

where K is the halting problem .
The prime model is $\mathfrak{M}_0$ clearly computable, and $\{s \in \omega : R(x,s)\}$ is finite or cofinite for all x.
The nonprime model $\mathfrak{M}_\alpha$ contains α many extra elements in $\bigcap_{s\in\omega} U_s$. Now fix any element $t \in \bigcap_{s\in\omega} U_s$.
Then for the unique $x \in U_{s+1} - U_s$, we have

$$R(x,t) \text{ iff } x \in K,$$

so the open diagram of any nonprime model can compute the halting problem K.

3. For any positive integer n, the language contains an $(n+2)$-ary relation R and the same unary relations U_s (for $s \in \omega$).

The prime model $\mathfrak{M}_0$ has universe ω where

$$R(x, s_0, \ldots, s_n) \text{ iff } x \in K_s$$
$$\text{for } s = \min\{s_0, \ldots, s_n\}$$
$$\text{and all } s_i \text{ pairwise distinct.}$$

Again, the nonprime model $\mathfrak{M}_\alpha$ contains α many extra elements in $\bigcap_{s\in\omega} U_s$, but it now takes n many elements in $\bigcap_{s\in\omega} U_s$ to compute the halting problem from the open diagram.

4. The language contains unary relations U_n (for each positive integer n) and k-ary relations $R_{k,s}$ (for each positive integer k and each integer s).

Given a Π_2^0-complete predicate

$$\forall n \exists m \varphi(k,n,m) \text{ (for computable } \varphi\text{)},$$

the prime model $\mathfrak{M}_0$ has universe ω (with the unary relations U_n as before) where $R_{k,s}$ is defined by

$$R_{k,s}(x_1,\ldots,x_k) \text{ iff } \forall n \leq s \exists m \leq j \varphi(k,n,m)$$

$$\text{(where } j = \min\{x_1,\ldots,x_k\}$$

$$\text{and all } x_i \text{ are pairwise distinct).}$$

Each nonprime model $\mathfrak{M}_\alpha$ contains α many extra elements in $\bigcap_{s\in\omega} U_s$. Thus $\bigcap_{s\in\omega} U_s$ is finite in each nonsaturated countable model, so each is computable. But the saturated model $\mathfrak{M}_\alpha$ is not computable since there $\forall n \exists m \varphi(n,m)$ iff

$$\exists \text{ distinct } y_1,\ldots,y_k \in \bigcap_{s\in\omega} U_s \ \forall s R_{k,s}(y_1,\ldots,y_k),$$

which is Σ_2^0 in the open diagram of $\mathfrak{M}_\alpha$.

5. The language contains binary relations R_n (for each positive integer n).
Given an infinite set $S \subseteq \omega$, the prime model $\mathfrak{M}_0$ has universe ω and is the disjoint union of, for each $n \in S$, one n-dimensional "cube" where R_m (for $m \leq n$) gives the adjacency relation between vertices along an edge in the mth dimension.
Each nonprime model $\mathfrak{M}_\alpha$ contains α many "ω-dimensional" cubes.
Call a function f <u>limitwise monotonic</u> if there is a computable function φ such that for each $x \in \omega$,

$$f(x) = \lim_s \varphi(x, s), \text{ and}$$

$$\varphi(x, s) \text{ is nondecreasing in } s.$$

Fix a Δ^0_2-set S which is not the range of a limitwise monotonic function (by an easy priority argument).
Then the corresponding prime model $\mathfrak{M}_0$ cannot be computable, else S would be the range of a limitwise monotonic function. But since S is Δ^0_2, each nonprime model $\mathfrak{M}_\alpha$ is computable.
(Discard unwanted finite cubes into any ω-cube.)

6. Add to the above construction of $\mathfrak{M}_1$ binary relations L_e (for each $e \in \omega$) and code K into each model $\mathfrak{M}_\alpha$ (for each $\alpha \geq 2$) by ensuring:

- $e \in K$ implies L_e is empty, and
- $e \notin K$ implies:

$$\mathfrak{M}_\alpha \models L_e(x, y) \text{ iff}$$

x, y each in cubes of sizes $\geq g(e)$ and
not together in subcube of size $< h(e)$
(where g is a computable function and
h is a Σ^0_1-function)

Now in $\mathfrak{M}_\alpha$ (for $\alpha \geq 2$), we can fix x, y in distinct ω-cubes, and we can now compute the halting problem K from $L_e(x, y)$.

Two Questions (Lempp, mid-1990's):

- Do the above results on spectra of computable models necessarily require an infinite language?

- If some model of an uncountably categorical first-order theory is computable, what can we say about its other countable models?
  Must the other countable models be
  - arithmetical?
  - $\mathbf{0}''$-computable?
  - $\mathbf{0}'$-computable?

A Related Question (Nies, Shore):

- How complicated can the spectrum of computable models be?

(Nies observed that a trivial upper bound is $\Sigma^0_{\omega+3}$.)