The logical strength and effective content of the Thin Set Theorem and the Free Set Theorem

Peter A. Cholak
Mariagnese Giusto
Jeffry L. Hirst
Carl G. Jockusch, Jr.

Definition. For $A \subseteq \mathbb{N}$, let $[A]^n$ denote the set of all increasing ordered n-tuples of elements of A (or n-element subsets of A).

The Thin Set Theorem. (Harvey Friedman) Let $f : [\mathbb{N}]^n \to \mathbb{N}$. Then there is an infinite set $A \subseteq \mathbb{N}$ such that A is f-*thin*, i.e. $f([A]^n) \neq \mathbb{N}$.

Let TS(n) denote the thin set theorem for exponent n. Let TS denote $\forall n \mathrm{TS}(n)$.

The thin set theorem is a starting point for Friedman's Boolean relation theory. It is an immediate consequence of Ramsey's theorem.

Questions:

1. Is the thin set theorem "equivalent" to Ramsey's theorem (level-by-level)?

2. Are thin sets "just as hard" to define or compute as homogeneous sets?

Let RT^n_k denote Ramsey's theorem for k-colorings of $[\mathbb{N}]^n$, i.e.

$$(\forall f : [\mathbb{N}]^n \to k)(\exists \text{ infinite } A \subseteq \mathbb{N}) \\ [f \text{ is constant on } [A]^n]$$

Theorem. (Jockusch, 1972)

(i) If $f : [\mathbb{N}]^n \to k$ is computable, there is an infinite Π^0_n set which is *f-homogeneous*, i.e. $f \restriction [A]^n$ is constant.

(ii) If $n \geq 2$, there is a computable $f : [\mathbb{N}]^n \to 2$ with no infinite Σ^0_n f-homogeneous set.

(iii) If $n \geq 2$, there is a computable $f : [\mathbb{N}]^n \to 2$ such that, for every infinite f-homogeneous set A, $0^{(n-2)} \leq_T A$.

Second order number theory

Language is the same as for first-order number theory except that it contains in addition second-order variables $X_0, X_1, \ldots$ and the membership symbol $\in$.

The base system is RCA_0 , whose axioms are the ordered semiring axioms, Σ^0_1 induction, and Δ^0_1-comprehension.

Example: $\mathrm{RCA}_0 \vdash \mathrm{RT}^1_2$.

A model of RCA_0 is obtained by letting the first-order part be standard and the second-order quantifiers range only over the *computable* sets.

Example: $\mathrm{RCA}_0 \nvdash \mathrm{RT}^2_2$.

Call $T \subseteq 2^{<\omega}$ a *tree* if T is closed downwards under extension.

A *path* through a tree $T \subseteq 2^{<\omega}$ is a function $f : \omega \to 2$ such that, for all $n \in \omega$, $f \restriction n \in T$.

Weak König's Lemma is the statement that every infinite tree $T \subseteq 2^{<\omega}$ has a path.

Weak König's Lemma is not provable in RCA_0 , since there is an infinite computable tree $T \subseteq 2^{<\omega}$ with no computable path.

The system WKL_0 is RCA_0 together with Weak König's Lemma.

An ω-model is a set $\mathcal{M} \subseteq 2^\omega$ and is identified with the second-order structure $\langle \mathcal{N}, \mathcal{M} \rangle$, where $\mathcal{N}$ is the standard model of first-order arithmetic and the second-order quantifiers range over $\mathcal{M}$.

There is an ω-model of WKL_0 consisting only of low sets A.

The system ACA_0 is RCA_0 together with arithmetic comprehension.

$\mathrm{ACA}_0 \vdash$ Weak König's Lemma

ACA_0 is strictly stronger than WKL_0 , and WKL_0 is strictly stronger than RCA_0 .

The first-order consequences of ACA_0 are exactly the theorems of Peano arithmetic.

The first-order consequences of RCA_0 are exactly the theorems of Peano arithmetic with induction restricted to (first-order) Σ_1 formulas.

RCA_0 and WKL_0 have the same Π^1_1 consequences (L. Harrington).

Reverse Math

Many mathematical statements (from analysis, algebra, combinatorics, ...) are provably equivalent over RCA_0 to one of the following:

(1) $0 = 0$

(2) Weak König's Lemma

(3) Arithmetical comprehension

(4) Arithmetical transfinite recursion

(5) Π^1_1-comprehension

Reverse Math and Calculus

(1) The Intermediate Value Theorem is provable in RCA_0 .

(2) The fact that every continuous function on $[0, 1]$ has a maximum is equivalent to Weak König's Lemma over RCA_0 .

(3) The fact that every Cauchy sequence of rational numbers converges is equivalent to arithmetic comprehension over RCA_0 .

Results on the strength of Ramsey's Theorem in second-order arithmetic. (See *Subsystems of Second Order Arithmetic*, by S. Simpson.)

(i) RT^2_2 is not provable in WKL_0 .

(ii) In RCA_0 , RT^n_k is provably equivalent to ACA_0 for each fixed $n \geq 3, k \geq 2$.

(iii) $(\forall n)\mathrm{RT}^n_2$ is not provable in ACA_0 .

Big Open Question: Does RT^2_2 imply Weak König's Lemma in RCA_0 ?

Theorem. (Seetapun, 1995). If $f : [\mathbb{N}]^2 \to 2$ and $C_0, C_1, \ldots$ are sets such that $(\forall i)[C_i \not\leq_T f]$, there is an infinite f-homogeneous set A such that $(\forall i)[C_i \not\leq_T A]$.

Corollary. (Seetapun). RT^2_2 is strictly weaker than ACA_0 over RCA_0 , and hence strictly weaker than RT^3_2 over RCA_0 .

Remark. For each n, RT^n_2 implies $\mathrm{TS}(n)$ in RCA_0 . Hence $\mathrm{TS}(n)$ is provable from ACA_0 , and $\mathrm{TS}(2)$ does not imply ACA_0 in RCA_0 .

Theorem.

(i) If $f : [\mathbb{N}]^n \to \mathbb{N}$ is computable, there is an infinite Π^0_n set which is f-thin.

(ii) If $n \geq 2$, there is a computable $f : [\mathbb{N}]^n \to \mathbb{N}$ with no infinite Σ^0_n thin set.

Corollary (H. Friedman)

(i) TS(2) is not provable in RCA_0 .

(ii) TS is not provable in ACA_0 .

Open Questions on Thin Sets

1. Does there exist a computable $f : [\mathbb{N}]^3 \to \mathbb{N}$ such that every infinite f-thin set has degree at least $\mathbf{0}'$?

(For Ramsey's theorem, the answer is "yes". Let $f(a, b, c) = 0$ if K_b and K_c have the same elements less than a, and $f(a, b, c) = 1$ otherwise. Every infinite f-homogeneous set computes $0'$.)

2. Is TS(3) equivalent to ACA_0 over RCA_0 ?

3. Is TS(2) equivalent to RT^2_2 over RCA_0 ?

The Free Set Theorem. (Harvey Friedman). Let $f : [\mathbb{N}]^n \to \mathbb{N}$. Then there is an infinite set $A \subseteq \mathbb{N}$ such that A is f-*free* , i.e.

for all $D \in [A]^n$, if $f(D) \in A$, then $f(D) \in D$.

Let FS(n) denote the free set theorem for exponent n, and let FS denote $(\forall n)$FS(n).

Remark.

$\mathrm{RCA}_0 \vdash (\forall n)[FS(n) \implies TS(n)]$.

Hence, FS(2) is not provable in RCA_0 , and FS is not provable in ACA_0 .

Corollary to a result on thin sets. For each $n \geq 2$ there is a computable function $f : [\mathbb{N}]^n \to \mathbb{N}$ such that no infinite Σ^0_n set is f-free.

Proof. Consider a computable function $f : [\mathbb{N}]^n \to \mathbb{N}$ such that no infinite Σ^0_n set is f-thin. Removing any one element from a free set produces a thin set, so no infinite Σ^0_n set is f-free.

Theorem. (H. Friedman)

$\mathrm{RCA}_0 \vdash (\forall n)[\mathrm{RT}^{n+1}_{n+2} \rightarrow \mathrm{FS}(n)]$.

Hence $\mathrm{ACA}_0 \vdash \mathrm{FS}(n)$ for each n.

Theorem.
$\mathrm{RCA}_0 \vdash (\forall n)[\mathrm{RT}^n_{2n+2} \to FS(n)]$.

The proof shows that for each computable $f : [\mathbb{N}]^n \to \mathbb{N}$ there is a computable $g : [\mathbb{N}]^n \to 2n+2$ such that for each infinite g-homogeneous set A there is an infinite f-free set $B \leq_T A$.

Theorem. For each computable $f : [\mathbb{N}]^n \to \mathbb{N}$ there is an infinite Π^0_n f-free set.

(Proved by analyzing the proof of the previous theorem and applying the technique used by Hummel and Jockusch (JSL '01) to show that each c.e. 2-coloring of $[\mathbb{N}]^n$ has an infinite Π^0_n homogeneous set.)

Theorem. (P. Cholak, C. Jockusch, and T. Slaman, JSL, 2001) Suppose that $n \geq 2$. Let $f : [\mathbb{N}]^n \to k$ be computable. Then there is an infinite f-homogeneous set A such that $A'' \leq_T 0^{(n)}$.

Corollary. If $n \geq 2$ and $f : [\mathbb{N}]^n \to \mathbb{N}$ is computable, then there is an infinite f-free set A such that $A'' \leq_T 0^{(n)}$, and hence there is also an infinite f-thin set with this property.

Theorem. (P. Cholak, C. Jockusch, and T. Slaman, JSL, 2001) The systems

$\mathrm{RCA}_0 + \Sigma_2$-induction

and

$\mathrm{RCA}_0 + \Sigma_2$-induction $+ \mathrm{RT}^2_2$

have the same Π^1_1 consequences.

Corollary. The systems

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Summary

1. The following four statements are all provable in RCA_0 :

(i) $(\forall n)[\mathrm{RT}^n_{2n+2} \implies \mathrm{FS}(n)]$

(ii) $(\forall n)[\mathrm{FS}(n) \implies \mathrm{TS}(n)]$

(iii) $(\forall n)[\mathrm{TS}(n+1) \implies \mathrm{TS}(n)]$

(iv) $(\forall n)[\mathrm{FS}(n+1) \implies \mathrm{FS}(n)]$

2. TS(2) is not provable in RCA_0 .

3. TS is not provable in ACA_0 .

4. For each n, the three statements RT^n_2, $\mathrm{FS}(n)$, and $\mathrm{TS}(n)$ are provable in ACA_0 and have similar effective bounds. (For the arithmetical hierarchy, $A \in \Pi^0_n$ is best possible for $n \geq 2$. For Turing degrees, get A with $A'' \leq_T 0^{(n)}$ for each $n \geq 2$.)

Open Questions

Does FS (or some $FS(n)$) imply ACA_0 in RCA_0 ?

Computability version: Does there exist a computable $f : [\mathbb{N}]^n \to \mathbb{N}$ such that every infinite f-free set A satisfies $K \leq_T A$?

Does FS(2) imply RT^2_2 in RCA_0 ?

Does $TS(n)$ imply $FS(n)$ in RCA_0 ?

Does TS imply FS(2) in RCA_0 ?

Theorem $\mathrm{WKL}_0 \nvdash TS(2)$.

Proof. There is a computable $f : [\mathbb{N}]^2 \to \mathbb{N}$ such that no infinite Σ^0_2 set is thin and hence no infinite low set is thin. Hence any ω model for WKL_0 consisting only of low sets fails to satisfy TS(2).

Open Questions

Do any of RT^2_2, TS, TS(2), FS, FS(2) imply Weak König's Lemma in RCA_0 ?

Definition Let $\mathbf{a}$ be a Turing degree. Then $\mathbf{a} >> \mathbf{0}$ means that every $\{0, 1\}$–valued computable partial function has a total $\mathbf{a}$-computable extension.

Is there a computable $f : [\mathbb{N}]^2 \to \mathbb{N}$ such that every infinite f-thin set has degree $\mathbf{a} >> \mathbf{0}$?

The analogous questions for TS, FS, FS(2) and RT^2_2 are also open.

A "consistency" result

Definition. A set A is free for a *partial* function ψ on $[\mathbb{N}]^n$ if there do not exist $x_1 < x_2 < \cdots < x_n$ with each x_i in A, and $\psi(x_1, \ldots, x_n) \downarrow\in A - \{x_1, \ldots, x_n\}$.

Corollary. The following result (*) holds *when relativized to* $0'$:

(*) For every computable partial function ψ on $[\mathbb{N}]^2$ there is an infinite Π^0_2 free set.

Proof. Suppose ψ is a $0'$-computable partial function defined on $[\mathbb{N}]^2$. Let g be a 3-place computable function so that $\psi(a, b) = \lim_s g(a, b, s)$ for all (a, b) in the domain of ψ. Let A be an infinite Π^0_3 g-free set. Then A is an infinite $\Pi^{0,K}_2$ ψ-free set.

(*) For every computable partial function ψ on $[\mathbb{N}]^2$ there is an infinite Π^0_2 free set.

Conclusion. (*) cannot be refuted by a relativizable argument.

Open Question. Is (*) true?

Remark. Two different proofs of (*) for the case where ψ is total break down when ψ is partial.

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