Herrmann's work on chains and antichains in computable partial orderings

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Preliminaries

Definition 1

- $(P, <_P)$ *is a* partial ordering *if* $<_P$ *is a transitive irreflexive relation on* P*.*
- *If* $(P, <_P)$ *is a partial ordering, a set* $C \subseteq P$ *is a* chain *if* C *is linearly ordered by* $<_P$*.*
- *If* $(P, <_P)$ *is a partial ordering, a set* $C \subseteq P$ *is an* antichain *if no two distinct elements of* C *are* $<_P$*-comparable.*

Theorem 2 Chain-Antichain Theorem (CAC) *Every infinite partially ordered set contains an infinite chain or an infinite antichain.*

Proof. Immediate from Ramsey's theorem for 2-colorings of pairs RT^2_2.

Let $[A]^k = \{D \subseteq A : |D| = k\}$.

RT^2_2: If $f : [\omega]^2 \to \{0, 1\}$, there is an infinite set $A \subseteq \omega$ which is f*-homogeneous*, i.e. f is constant on $[A]^2$.

Arithmetical Hierarchy

Definition 3 *$A \subseteq \omega$ is Σ^0_2 if there exists a computable R such that, for all a,*

$$a \in A \iff (\exists b)(\forall c) R(a, b, c)$$

Note: The following are equivalent:

- A is Σ^0_2
- A is c.e. in K
- For some computable f, $A = \{n : \lim_s f(n, s) = 1\}$.

A is Π^0_2 iff $\overline{A} = \omega - A$ is Σ^0_2.

A is Δ^0_2 iff A is both Σ^0_2 and Π^0_2 iff $A \leq_T K$.

Theorem 4 Π^0_2 **- CAC :** *Let* $<_P$ *be a computable partial ordering of* ω. *Then* $<_P$ *has either an infinite* Π^0_2 *chain or an infinite* Π^0_2 *antichain. In fact, if* $<_P$ *has no infinite* Δ^0_2 *chain or antichain, it has both an infinite* Π^0_2 *chain and an infinite* Π^0_2 *antichain.*

Proof. Use $\Pi^0_2 - RT^2_2$: If $f : [\omega]^2 \to \{0, 1\}$ is computable, there is an infinite Π^0_2 f-homogeneous set. In fact, ...

Question Does every computable partial ordering of ω have an infinite Σ^0_2 chain or antichain?

Note: $\Sigma^0_2 - RT^2_2$ is false.

Theorem 5 *(E. Herrmann). There is a computable partial ordering* $(\omega, <_P)$ *with no infinite* Σ^0_2 *chains or antichains.*

Outline of proof:

1. Define a computable partial ordering $<_u$ of ω with special properties.
2. Let $S_0, S_1, \ldots$ be a uniformly Σ^0_2 list of the Σ^0_2 chains and antichains of $<_u$.
3. **Main Step.** Construct an infinite computable set R with $R \cap S_e$ finite for all e.
4. The desired computable partial ordering $(\omega, <_P)$ is an effective copy of $(R, <_u)$.

Definition of $<_u$: Deferred

To list the Σ^0_2 chain or antichain S_e with a K oracle, list W_e^K until the first time, if ever, an element x appears in W_e^K so that $S_e^s \cup \{x\}$ is neither a chain nor an antichain for $<_u$, where S_e^s is the set of numbers already enumerated in S_e. Then let $S_e = S_e^s$, and if no such x exists, let $S_e = W_e^K$.

Requirements for Main Step:

$P_e : \quad |R| \geq e$

$N_e : \quad S_e \cap R$ is finite.

Theorem 6 *(E. Herrmann) Suppose that* $<_P$ *is a computable partial ordering of* ω *which has no infinite* Σ^0_2 *chains or antichains. Suppose also that* $<_L$ *is a computable* linear *ordering of* ω *which extends* $<_P$*. Then* $<_L$ *has order type* $\omega + (\omega^* + \omega) \cdot \eta + \omega^*$*, where* η *is the order-type of the rationals.*

Lemma 7 *(E. Hermann, J. Mileti) Let* $<_P$ *be a computable partial ordering with no infinite* Σ^0_2 *chains or antichains, and let* $<_L$ *be a computable linear ordering extending* P*. Then* $<_L$ *has no adjacent blocks.*

Lemma 8 *(J. Mileti) Suppose that* $<_P$ *is a computable partial ordering of* ω *and, for all* $a \in \omega$*, exactly one of the following two sets is infinite:*

$$\{b : a <_P b\} \quad , \quad \{b : b <_P a\}.$$

Then $<_P$ *has an infinite* Σ^0_2 *chain or antichain.*

Hirschfeldt and Shore used Herrmann's theorem to obtain the following Corollary:

Corollary 9 *(Hirschfeldt-Shore) There is a computable linear ordering of* ω *with no low subordering of type* ω, ω^* *or* $\omega + \omega^*$.

Recall that CAC is the statement in second-order arithmetic that every infinite partial order has an infinite chain or an infinite antichain.

Let RT^2_2 be Ramsey's theorem for 2-colorings of pairs.

Clearly RT^2_2 implies CAC in RCA_0, the base system for reverse mathematics.

The reverse implication is open.

Let WKL_0 be RCA_0 together with Weak König's Lemma.

WKL_0 has an ω-model consisting only of low sets.

To show that a combinatorial theorem $(\forall X)(\exists Y)\varphi(X, Y)$ is not provable from WKL_0 it suffices to give a computable instance X for which there is no low solution Y.

Corollary 10 *CAC is not provable in* WKL_0.

Corollary 11 *(Hirschfeldt-Shore) It is not provable in* WKL_0 *that every infinite linear ordering has a subordering of type* ω, ω^*, *or* $\omega + \omega^*$.

For this and many related results, see [2].

REFERENCES

[1] E. Herrmann, Infinite chains and antichains in computable partial orderings, J. Symbolic Logic 66 (2001), 923–934.

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[3] C. Jockusch, Ramsey's theorem and recursion theory, J. Symbolic Logic 37 (1972), 268–280.

[4] J. Mileti, Unpublished lemma, seminar, and many helpful conversations!