Herrmann’s work on chains and antichains in computable partial orderings

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Definition 1 • \((P, <_P)\) is a partial ordering if \(<_P\) is a transitive irreflexive relation on \(P\).

• If \((P, <_P)\) is a partial ordering, a set \(C \subseteq P\) is a chain if \(C\) is linearly ordered by \(<_P\).

• If \((P, <_P)\) is a partial ordering, a set \(C \subseteq P\) is an antichain if no two distinct elements of \(C\) are \(<_P\)-comparable.
Theorem 2 Chain-Antichain Theorem (CAC) Every infinite partially ordered set contains an infinite chain or an infinite antichain.

Proof. Immediate from Ramsey’s theorem for 2-colorings of pairs $RT_2^2$.

Let $[A]^k = \{D \subseteq A : |D| = k\}$.

$RT_2^2$: If $f : [\omega]^2 \to \{0, 1\}$, there is an infinite set $A \subseteq \omega$ which is $f$-homogeneous, i.e. $f$ is constant on $[A]^2$. 
Arithmetical Hierarchy

**Definition 3** \( A \subseteq \omega \) is \( \Sigma^0_2 \) if there exists a computable \( R \) such that, for all \( a \),

\[
a \in A \iff (\exists b)(\forall c)R(a, b, c)
\]

**Note:** The following are equivalent:

- \( A \) is \( \Sigma^0_2 \)
- \( A \) is c.e. in \( K \)
- For some computable \( f \),
  \[
  A = \{ n : \lim_s f(n, s) = 1 \}.
  \]

\( A \) is \( \Pi^0_2 \) iff \( \overline{A} = \omega - A \) is \( \Sigma^0_2 \).

\( A \) is \( \Delta^0_2 \) iff \( A \) is both \( \Sigma^0_2 \) and \( \Pi^0_2 \) iff \( A \leq_T K \).
Theorem 4 Π₂⁰ - CAC: Let <ₚ be a computable partial ordering of ω. Then <ₚ has either an infinite Π₂⁰ chain or an infinite Π₂⁰ antichain. In fact, if <ₚ has no infinite Δ₂⁰ chain or antichain, it has both an infinite Π₂⁰ chain and an infinite Π₂⁰ antichain.

Proof. Use Π₂⁰ – RT₂²: If 
f : [ω]² → {0, 1} is computable, there is an infinite Π₂⁰ f-homogeneous set. In fact, . . . .

Question Does every computable partial ordering of ω have an infinite Σ₂⁰ chain or antichain?

Note: Σ₂⁰ – RT₂² is false.
Theorem 5 (E. Herrmann). There is a computable partial ordering $(\omega, <_P)$ with no infinite $\Sigma^0_2$ chains or antichains.

Outline of proof:

1. Define a computable partial ordering $<_u$ of $\omega$ with special properties.

2. Let $S_0, S_1, \ldots$ be a uniformly $\Sigma^0_2$ list of the $\Sigma^0_2$ chains and antichains of $<_u$.

3. **Main Step.** Construct an infinite computable set $R$ with $R \cap S_e$ finite for all $e$.

4. The desired computable partial ordering $(\omega, <_P)$ is an effective copy of $(R, <_u)$. 
Definition of $<_u$ : Deferred

To list the $\Sigma_2^0$ chain or antichain $S_e$ with a $K$ oracle, list $W_e^K$ until the first time, if ever, an element $x$ appears in $W_e^K$ so that $S'_e \cup \{x\}$ is neither a chain nor an antichain for $<_u$, where $S'_e$ is the set of numbers already enumerated in $S_e$. Then let $S_e = S'_e$, and if no such $x$ exists, let $S_e = W_e^K$.

Requirements for Main Step:

$P_e : \quad |R| \geq e$

$N_e : \quad S_e \cap R$ is finite.
Theorem 6 (E. Herrmann) Suppose that $<_P$ is a computable partial ordering of $\omega$ which has no infinite $\Sigma^0_2$ chains or antichains. Suppose also that $<_L$ is a computable linear ordering of $\omega$ which extends $<_P$. Then $<_L$ has order type $\omega + (\omega^* + \omega) \cdot \eta + \omega^*$, where $\eta$ is the order-type of the rationals.
Lemma 7 (E. Hermann, J. Mileti) Let $<_P$ be a computable partial ordering with no infinite $\Sigma^0_2$ chains or antichains, and let $<_L$ be a computable linear ordering extending $P$. Then $<_L$ has no adjacent blocks.
Lemma 8 (J. Mileti) Suppose that $<_P$ is a computable partial ordering of $\omega$ and, for all $a \in \omega$, exactly one of the following two sets is infinite:

$$\{ b : a <_P b \} , \quad \{ b : b <_P a \} .$$

Then $<_P$ has an infinite $\Sigma^0_2$ chain or antichain.
Hirschfeldt and Shore used Herrmann’s theorem to obtain the following Corollary:

Corollary 9 (Hirschfeldt-Shore) There is a computable linear ordering of $\omega$ with no low subordering of type $\omega$, $\omega^*$ or $\omega + \omega^*$. 
Recall that CAC is the statement in second-order arithmetic that every infinite partial order has an infinite chain or an infinite antichain.

Let $\text{RT}_2^2$ be Ramsey’s theorem for 2-colorings of pairs.

Clearly $\text{RT}_2^2$ implies CAC in $\text{RCA}_0$, the base system for reverse mathematics.

The reverse implication is open.
Let WKL$_0$ be RCA$_0$ together with Weak König’s Lemma.

WKL$_0$ has an $\omega$-model consisting only of low sets.

To show that a combinatorial theorem $(\forall X)(\exists Y)\varphi(X,Y)$ is not provable from WKL$_0$ it suffices to give a computable instance $X$ for which there is no low solution $Y$. 
Corollary 10  *CAC* is not provable in *WKL*$_0$.

Corollary 11  (*Hirschfeldt-Shore*) *It is not provable in* *WKL*$_0$ *that every infinite linear ordering has a subordering of type* $\omega$, $\omega^*$, *or* $\omega + \omega^*$.

*For this and many related results, see [2].*
REFERENCES


